

Nonautonomous mixed mKdV-sinh-Gordon hierarchy

J.F. Gomes, G. R. de Melo, L.H. Ymai and A.H. Zimerman

Instituto de Física Teórica-UNESP
Rua Dr Bento Teobaldo Ferraz 271, Bloco II,
01140-070, São Paulo, Brazil

Abstract

The construction of a nonautonomous mixed mKdV/sine-Gordon model is proposed by employing an infinite dimensional affine Lie algebraic structure within the zero curvature representation. A systematic construction of soliton solutions is provided by an adaptation of the dressing method which takes into account arbitrary time dependent functions. A particular choice of those arbitrary functions provides an interesting solution describing the transition of a pure mKdV system into a pure sine-Gordon soliton.

1 Introduction

Sometime ago, the study of nonlinear effects in lattice dynamics under the influence of a weak dislocation potential has lead to a mixed mKdV/sine-Gordon equation [1]. The system was shown to admit multisoliton solutions and an infinite set of conservation laws [1]. More recently the two-breather solution was discussed in [2] in connection with the propagation of few cycle pulses (FCP) in non linear optical media. According to ref. [2] the general mKdV/sine-Gordon equation, in fact, describes the propagation of a ultrashort optical pulses in a Kerr media. Moreover, it was shown in [3] that, when the ressonance frequency of atoms in the physical system are well above or well below the characteristic duration of the pulse, the propagation is described by the mKdV or sine-Gordon equations respectively. The main object of this paper is to provide a systematic construction of soliton solutions that describe the *transition between the two regimes*, i.e. governed by the mKdV and sine-Gordon equations. This is accomplished by considering the mixed integrable model proposed in [1] with two arbitrary time-dependent

coefficients. In this paper we show the integrability of the mixed model with time dependent coefficients and that, by suitable choice of these coefficients as a smooth step-type functions (as shown in figs. 1 and 3) we obtain exact solutions for the mKdV-SG transition and hence a more realistic description of such phenomena.

2 Algebraic Formalism

In ref. [4] the algebraic structure of the mixed mKdV/sine-Gordon equation was formulated within the zero curvature representation and a graded infinite dimensional Lie algebraic structure as we shall now briefly review. Consider the associated $\mathcal{G} = sl(2)$ Lie algebra with generators satisfying $[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$, $[E_{\alpha}, E_{-\alpha}] = h$ and grading operator $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$. Q decomposes the associated affine Lie algebra $\hat{sl}(2)$ into graded subspaces, $\hat{\mathcal{G}} = \oplus_i \mathcal{G}_i$,

$$\mathcal{G}_{2m} = \{\lambda^m h\}, \quad \mathcal{G}_{2m+1} = \{E_+^{(2m+1)} \equiv \lambda^m (E_{\alpha} + \lambda E_{-\alpha}), E_-^{(2m+1)} \equiv \lambda^m (E_{\alpha} - \lambda E_{-\alpha})\}, \quad (1)$$

$m = 0, \pm 1, \pm 2, \dots$ In [4], a simple proof that a mixed mKdV/sine-Gordon hierarchy is indeed an integrable model follows from the zero curvature representation of the integrable hierarchy generated by

$$[\partial_x + E_+^{(1)} + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + \dots D^{(0)} + D^{(-1)}] = 0, \quad (2)$$

where $D^{(j)} \in \mathcal{G}_j$ and $A_0 = vh$ contains the field variable $v = v(x, t)$. According to the subspace decomposition (2) for $N = 3$ and $t = t_3$ which corresponds to the mixed mKdV-SG equation.

Let us parametrize,

$$\begin{aligned} D^{(3)} &= a_3 E_+^{(3)} + b_3 E_-^{(3)}, & D^{(2)} &= c_2 \lambda h, \\ D^{(1)} &= a_1 E_+^{(1)} + b_1 E_-^{(1)}, & D^{(0)} &= c_0 h, \\ D^{(-1)} &= a_{-1} E_+^{(-1)} + b_{-1} E_-^{(-1)}. \end{aligned} \quad (3)$$

The grade by grade decomposing of eqn. (2) leads to

$$b_3 = 0, \quad \partial_x a_3 = 0, \quad c_2 = a_3 v, \quad b_1 = \frac{1}{2} \partial_x c_2, \quad (4)$$

$$\partial_x a_1 + 2v b_1 = 0, \quad \partial_x b_1 + 2v a_1 - 2c_0 = 0, \quad (5)$$

$$\partial_x a_{-1} + 2v b_{-1} = 0, \quad \partial_x b_{-1} + 2v a_{-1} = 0, \quad (6)$$

together with the equation of motion

$$\partial_x c_0 - \partial_t v - 2b_{-1} = 0. \quad (7)$$

In solving eqns. (4) we find

$$a_3 = a_3(t), \quad b_1 = \frac{a_3(t)}{2} v_x, \quad c_2 = a_3(t) v, \quad (8)$$

where $a_3(t)$ is an arbitrary function of t . Introducing (8) in the first eqn. (5), we obtain

$$\partial_x \left(a_1 + a_3(t) \frac{v^2}{2} \right) = 0,$$

which implies that

$$a_1 + a_3(t) \frac{v^2}{2} = f_1(t),$$

where $f_1(t)$ is another arbitrary function of t . It therefore follows that

$$a_1 = f_1(t) - a_3(t) \frac{v^2}{2}. \quad (9)$$

Substituting (9) in the second eqn. (5), we get

$$c_0 = \frac{a_3(t)}{4} (v_{xx} - 2v^3) + f_1(t) v. \quad (10)$$

Adding and subtracting eqns. (6), we obtain

$$\partial_x a_{\pm} = \mp 2v a_{\pm}, \quad (11)$$

where we have denoted

$$a_{\pm} = a_{-1} \pm b_{-1}.$$

Without loss of generality we may solve (11) by changing the variable

$$vh = -\partial_x BB^{-1} = \phi_x h, \quad B = e^{-\phi h}, \quad (12)$$

which leads us to $a_{\pm} = f_{-1}(t)e^{\mp 2\phi}$, where $f_{-1}(t)$ is another arbitrary function of t . Writing

$$a_{-1} = \frac{a_+ + a_-}{2}, \quad b_{-1} = \frac{a_+ - a_-}{2},$$

we find

$$a_{-1} = f_{-1}(t) \cosh(2\phi), \quad b_{-1} = -f_{-1}(t) \sinh(2\phi). \quad (13)$$

Substituting (10), (12) and (13) in (7), we finally obtain

$$\frac{a_3(t)}{4} (\phi_{xxxx} - 6\phi_x^2 \phi_{xx}) + f_1(t) \phi_{xx} - \phi_{xt} + 2f_{-1}(t) \sinh(2\phi) = 0. \quad (14)$$

Considering $f_1(t) = 0$, $a_3(t) = \text{constant}$ and $f_{-1}(t) = \text{constant}$ we find the usual mixed mKdV/sine-Gordon equation. For $f_1(t) = 0$, $a_3(t)$ a given numerical constant $\neq 0$ we recover eqn. (10) of ref. [5]. Moreover for $f_{-1}(t) = 0$, we recover equation considered in [6] with a choice of coefficients that makes the model integrable.

We should point out that by change of coordinates (see for instance [7]) $(x, t) \rightarrow (\tilde{x}, \tilde{t}) = (x + V(t), t)$ where $V_t = f_1(t)$ followed by a subsequently change $\tilde{t} \rightarrow T = \int a_3(\tilde{t}) d\tilde{t}$ and re-scaling $f_{-1} \rightarrow \tilde{\eta}$ leads to

$$\frac{1}{4} (\phi_{xxxx} - 6\phi_x^2 \phi_{xx}) - \phi_{xt} + 2\tilde{\eta}(t) \sinh(2\phi) = 0. \quad (15)$$

Although eqn. (15) corresponds to the equation discussed in [5] the object of this paper is to consider a class of solutions that interpolates between the mKdV and sine-Gordon equations.

This is more conveniently accomplished by employing eqn. (14) where the two arbitrary functions $a_3(t)$ and $f_{-1}(t)$ (with $f_1(t) = 0$) can be chosen as step-like limiting functions (Figs. 1 and 3) as we shall see.

3 Construction of Soliton Solutions

In order to construct, in a systematic manner, the soliton solutions of the mixed model let us now recall some basic aspects of the dressing method (see for instance [8]). The zero curvature representation (2) implies in a pure gauge configuration, i.e.,

$$\partial_x + E + A_0 = \partial_x TT^{-1}, \quad \partial_t + D^{(3)} + \dots D^{(-1)} = \partial_t TT^{-1}, \quad (16)$$

In particular, the vacuum is obtained by setting $\phi_{vac} = 0$ ¹ which implies,

$$\partial_x T_0 T_0^{-1} = -E_+^{(1)}, \quad \partial_t T_0 T_0^{-1} = -a_3(t)E_+^{(3)} - f_{-1}(t)E_+^{(-1)} - f_1(t)E_+^{(1)}. \quad (17)$$

which after integration yields

$$T_0 = \exp \left(- \int^t dt' a_3(t') E_+^{(3)} - \int^t dt' f_{-1}(t') E_+^{(-1)} - \int^t dt' f_1(t') E_+^{(1)} \right) \exp(-x E_+^{(1)}), \quad (18)$$

Following the dressing method explained in [8] and employed in [4] we define the tau-functions

$$\tau_n \equiv \langle \lambda_n | B | \lambda_n \rangle = \langle \lambda_n | T_0 g T_0^{-1} | \lambda_n \rangle, \quad (19)$$

where $\lambda_n, n = 0, 1$ are fundamental weights of the full affine Kac-Moody algebra $\hat{sl}(2)$, g is a constant group element which classifies the soliton solutions and B is a zero grade group

¹For a general member of the hierarchy evolving according $t = t_{2n+1}$, the vacuum configuration implies

$$\partial_x T_0 T_0^{-1} = -E_+^{(1)}, \quad \partial_{t_{2n+1}} T_0 T_0^{-1} = -a_{2n+1}(t) E_+^{(2n+1)} - f_{-1}(t) E_+^{(-1)} - \sum_{k=1}^n f_{2k-1}(t) E_+^{(2k-1)}.$$

element containing the physical fields. In order to ensure heighest weight representations we now introduce central extensions within the affine Lie algebra, characterized by \hat{c} , i.e.,

$$\begin{aligned} [h^{(n)}, E_{\pm\alpha}^{(m)}] &= \pm 2E_{\pm\alpha}^{(n+m)}, \\ [E_{\alpha}^{(n)}, E_{-\alpha}^{(m)}] &= h^{(n+m)} + n\delta_{n+m,0}\hat{c}, \\ [h^{(n)}, h^{(m)}] &= 2n\delta_{n+m,0}\hat{c}, \end{aligned}$$

and define highest weight representations, i.e.,

$$h|\lambda_n\rangle = \delta_{n,1}|\lambda_n\rangle, \quad \hat{c}|\lambda_n\rangle = |\lambda_n\rangle, \quad \mathcal{G}_i|\lambda_n\rangle = 0, \quad i > 0, \quad (20)$$

$n = 0, 1$. Under this affine picture the group element B acquires a central term contribution,

$$B = e^{-\phi h} e^{-\nu \hat{c}}. \quad (21)$$

In order to obtain explicit space-time dependence from the r.h.s. of (19) we consider the vertex operators,

$$V(\gamma) = \sum_{n=-\infty}^{\infty} (\lambda^n h - \frac{1}{2}\hat{c}\delta_{n,0})\gamma^{-2n} + E_-^{(2n+1)}\gamma^{-2n-1}, \quad (22)$$

satisfying

$$[E_+^{(2n+1)}, V(\gamma)] = -2\gamma^{2n+1}V(\gamma). \quad (23)$$

For a general M -soliton solution the group element g in (19) is written as

$$g = \prod_{j=1}^M e^{\alpha_j V(\gamma_j)}, \quad (24)$$

where α_j are arbitrary constants. We therefore obtain

$$\begin{aligned} \tau_0 &= e^{-\nu} = \langle \lambda_0 | \prod_{j=1}^M e^{\alpha_j \rho_j(x,t) V(\gamma_j)} | \lambda_0 \rangle, \\ \tau_1 &= e^{-\phi-\nu} = \langle \lambda_1 | \prod_{j=1}^M e^{\alpha_j \rho_j(x,t) V(\gamma_j)} | \lambda_1 \rangle, \end{aligned}$$

where ²

$$\rho_j(x, t) = e^{2\gamma_j x + 2\gamma_j^3 A_3(t) + 2\gamma_j F_1(t) + 2\gamma_j^{-1} A_{-1}(t)}, \quad (25)$$

$$A_3(t) = \int dt a_3(t), \quad F_1(t) = \int dt f_1(t), \quad A_{-1}(t) = \int dt f_{-1}(t).$$

As an illustrative example, we consider the one and two soliton cases, $M = 1, 2$, where

$$\tau_0^{1-sol} = e^{-\nu} = 1 - \frac{\alpha_1}{2}\rho_1, \quad \tau_1^{1-sol} = e^{-\nu-\phi} = 1 + \frac{\alpha_1}{2}\rho_1, \quad (26)$$

and

$$\begin{aligned} \tau_0^{2-sol} &= e^{-\nu} = 1 - \frac{\alpha_1}{2}\rho_1 - \frac{\alpha_2}{2}\rho_2 + \alpha_1\alpha_2 A_{1,2}\rho_1\rho_2, \\ \tau_1^{2-sol} &= e^{-\nu-\phi} = 1 + \frac{\alpha_1}{2}\rho_1 + \frac{\alpha_2}{2}\rho_2 + \alpha_1\alpha_2 A_{1,2}\rho_1\rho_2, \end{aligned} \quad (27)$$

respectively. In order to obtain (26) and (27) where we have used the fact that

$$\begin{aligned} \langle \lambda_n | V(\gamma) | \lambda_n \rangle &= \delta_{n,1} - \frac{1}{2}, \quad n = 0, 1 \\ \langle \lambda_n | V(\gamma_1)V(\gamma_2) | \lambda_n \rangle &= A_{1,2} = \frac{1}{4} \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \right)^2. \end{aligned}$$

The general two soliton solution can then be written as

$$\phi = \ln \left(\frac{\tau_0}{\tau_1} \right) = \ln \left(\frac{1 - \frac{\alpha_1}{2}\rho_1 - \frac{\alpha_2}{2}\rho_2 + \alpha_1\alpha_2 A_{1,2}\rho_1\rho_2}{1 + \frac{\alpha_1}{2}\rho_1 + \frac{\alpha_2}{2}\rho_2 + \alpha_1\alpha_2 A_{1,2}\rho_1\rho_2} \right), \quad (28)$$

while the one soliton is obtained from (28) by setting $\alpha_2 = 0$.

² In considering a general $(2n+1)$ -th member of the hierarchy,

$$\rho_j(x, t) = e^{2\gamma_j x + 2\gamma_j^{2n+1} A_{2n+1}(t) + 2 \sum_{k=1}^n \gamma_j^{2k-1} F_{2k-1}(t) + 2\gamma_j^{-1} A_{-1}(t)}, \quad A_{2n+1}(t) = \int dt a_{2n+1}(t), \quad F_{2k-1} = \int dt f_{2k-1}(t).$$

4 Applications

According to ref. [2] the propagation of a FCP with frequency ω on a dielectric media with characteristic frequency Ω , $\omega \ll \Omega$ can be described by the mKdV equation

$$\phi_{z\tau} + a \left(\frac{3}{2} \phi_\tau^2 \phi_{\tau\tau} + \phi_{\tau\tau\tau\tau} \right) = 0,$$

where the coordinates z and τ correspond respectively to the propagation distance and retarded time, while the electric field $E = \phi_\tau$. For the case where $\omega \gg \Omega$, the system is described by the sine-Gordon equation,

$$\phi_{z\tau} - b \sin \phi = 0.$$

If we now consider a dielectric media with two characteristic frequencies Ω_1, Ω_2 , the case in the regime $\Omega_1 \ll \omega \ll \Omega_2$, is described by the mixed mKdV-SG equation ,

$$\phi_{z\tau} + a \left(\frac{3}{2} \phi_\tau^2 \phi_{\tau\tau} + \phi_{\tau\tau\tau\tau} \right) - b \sin \phi = 0,$$

where the two constants a and b are related to the non-linear and dispersion properties of the media.

In order to adapt model (14) to such situation, define

$$a_3(t) = -4a\theta_1(t), \quad f_1(t) = 0, \quad f_{-1}(t) = \frac{b}{4}\theta_2(t),$$

re-scaling $\phi \rightarrow \frac{i}{2}\phi$, $t \rightarrow z$, $x \rightarrow \tau$, eqn. (14) becomes

$$a \theta_1(z) \left(\phi_{\tau\tau\tau\tau} + \frac{3}{2} \phi_\tau^2 \phi_{\tau\tau} \right) + \phi_{z\tau} - b \theta_2(z) \sin \phi = 0. \quad (29)$$

If we now substitute $\alpha_k \rightarrow -2i\alpha_k$, and make use of the identity

$$\arctan X = \frac{1}{2i} \ln \left(\frac{1+iX}{1-iX} \right),$$

we find that the two soliton solution (28) may be written as

$$\phi = 4 \arctan \left(\frac{\alpha_1 \rho_1 + \alpha_2 \rho_2}{1 - 4\alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2} \right), \quad (30)$$

where

$$\rho_j = \exp \left(2\gamma_j \tau + 2\gamma_j^3 A_3(z) + 2\gamma_j^{-1} A_{-1}(z) \right),$$

$$A_3(z) = -4a \int^z dz' \theta_1(z'), \quad A_{-1}(z) = \frac{b}{4} \int^z dz' \theta_2(z'). \quad (31)$$

4.1 Transition mKdV-SG

Consider now a dielectric media in which $\Omega_1 \gg 2\gamma_j$ in the region $z < z_1$ and $\Omega_2 \ll 2\gamma_j$ in the region $z > z_2$ with $z_1 > z_2$ such that there exist an overlap region in which the media admits the two characteristic frequencies, $\Omega_2 \ll 2\gamma_j \ll \Omega_1$. As an example to describe such a realistic situation, we take both $\theta_1(z)$ and $\theta_2(z)$ as step-like functions (see fig. 1) with

$$\theta_1(z) = \frac{1}{2} - \frac{1}{\pi} \arctan [\beta_1(z - z_1)], \quad \theta_2(z) = \frac{1}{2} + \frac{1}{\pi} \arctan [\beta_2(z - z_2)],$$

where β_1 e β_2 are phenomenological parameters describing the transition between the two medias (see fig. 1).

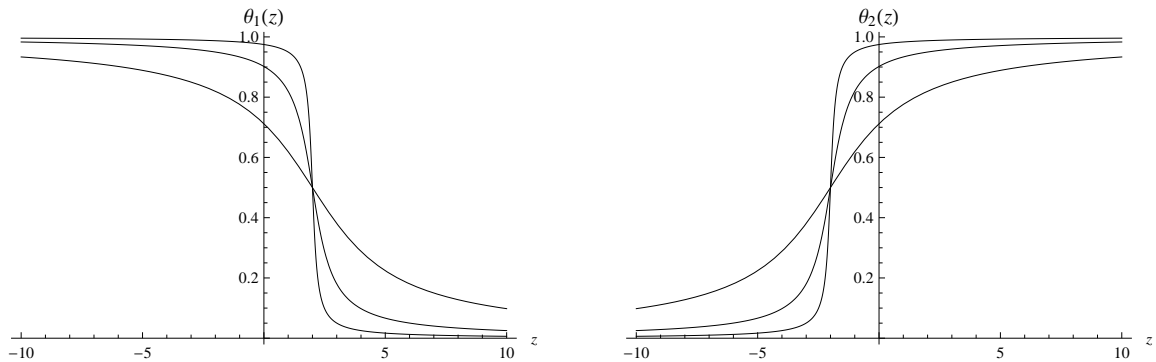


Figure 1: Plots of $\theta_1(z)$ and $\theta_2(z)$ for $\beta_1 = \beta_2 = \{\frac{\pi}{8}, \frac{\pi}{2}, 2\pi\}$, with $z_1 = 2$ and $z_2 = -2$.

They can therefore be integrated from (32) to yield,

$$\begin{aligned} -\frac{1}{4a}A_3(z) &= \frac{z}{2} - \frac{(z - z_1)}{\pi} \arctan[\beta_1(z - z_1)] + \frac{1}{2\pi\beta_1} \ln[1 + \beta_1^2(z - z_1)^2], \\ \frac{4}{b}A_{-1}(z) &= \frac{z}{2} + \frac{(z - z_2)}{\pi} \arctan[\beta_2(z - z_2)] - \frac{1}{2\pi\beta_2} \ln[1 + \beta_2^2(z - z_2)^2]. \end{aligned}$$

The $\beta_1, \beta_2 \rightarrow +\infty$ limit, for $z_1 = z_2 = 0$ correspond to the system governed by the pure mKdV in the region $z < 0$ and by the pure sine-Gordon equation for $z > 0$. In such limit, we have,

$$\theta_1(z) = \frac{1}{2} \left(1 - \frac{|z|}{z} \right), \quad \theta_2(z) = \frac{1}{2} \left(1 + \frac{|z|}{z} \right), \quad (32)$$

and

$$-\frac{1}{4a}A_3(z) = \frac{1}{2}(z - |z|), \quad \frac{4}{b}A_{-1}(z) = \frac{1}{2}(z + |z|). \quad (33)$$

Figure 2 below shows the transition mKdV-SG for the one soliton solution, $\alpha_1 = 1, \alpha_2 = 0$. The plot on the right shows the soliton solution viewed from the above and displays the transition (different velocities) from the mKdV to the SG solitons.

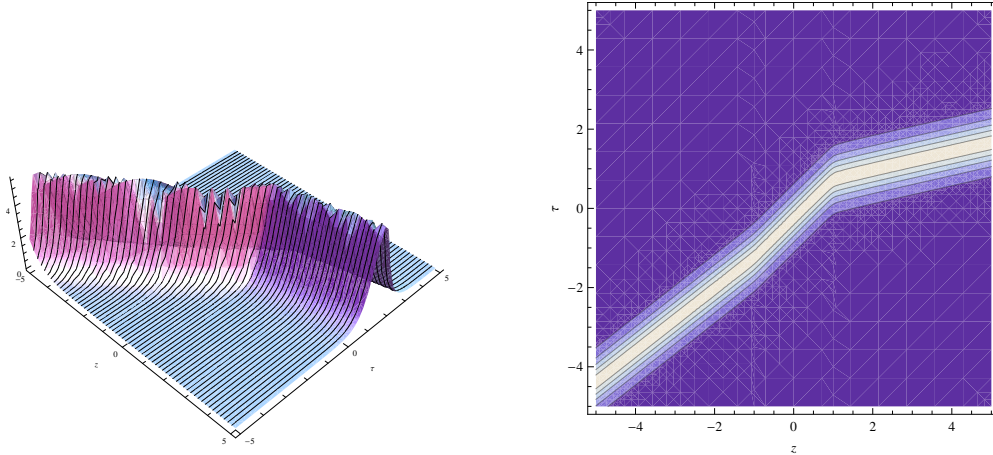


Figure 2: Plot (left) and contour plot (right) of ϕ_τ with $a = \frac{1}{10}$, $b = -\frac{7}{4}$, $\beta_1 = \beta_2 = 8\pi$, $z_1 = 1$, $z_2 = -1$ and $\gamma_1 = \frac{14}{10}$.

4.2 Transition mKdV-SG-mKdV

Another example consist in two equal media separated by a second one describing, for instance the mKdV-SG-mKdV transition. Mathematically this situation may be described by combining theta-type functions (see fig. 3), i.e.,

$$\begin{aligned}\theta_1(z) &= 1 - \frac{1}{\pi} \arctan \left[\bar{\beta}_1(z - \bar{z}_1) \right] + \frac{1}{\pi} \arctan \left[\bar{\beta}_2(z - \bar{z}_2) \right], & \bar{z}_1 < \bar{z}_2, \\ \theta_2(z) &= \frac{1}{\pi} \arctan [\beta_1(z - z_1)] - \frac{1}{\pi} \arctan [\beta_2(z - z_2)], & z_1 < z_2,\end{aligned}$$

with $z_1 \leq \bar{z}_1$, and $\bar{z}_2 \leq z_2$ to guarantee the existence of an overlap region between the two medias.

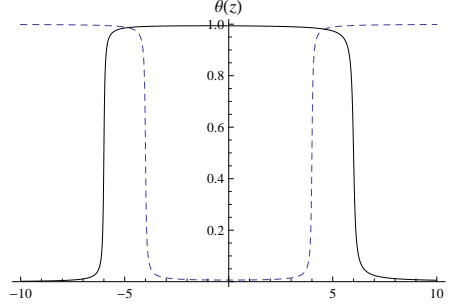


Figure 3: Plot of $\theta_1(z)$ (dashed) and $\theta_2(z)$ (continuum) with $\beta_1 = \beta_2 = \bar{\beta}_1 = \bar{\beta}_2 = 8\pi$, $z_1 = -6$, $z_2 = 6$, $\bar{z}_1 = -4$ and $\bar{z}_2 = 4$.

After integration (32) we find

$$\begin{aligned}-\frac{1}{4a}A_3(z) &= z - \frac{(z - \bar{z}_1)}{\pi} \arctan \left[\bar{\beta}_1(z - \bar{z}_1) \right] + \frac{(z - \bar{z}_2)}{\pi} \arctan \left[\bar{\beta}_2(z - \bar{z}_2) \right] \\ &\quad + \frac{1}{2\pi\bar{\beta}_1} \ln \left[1 + \bar{\beta}_1^2(z - \bar{z}_1)^2 \right] - \frac{1}{2\pi\bar{\beta}_2} \ln \left[1 + \bar{\beta}_2^2(z - \bar{z}_2)^2 \right], \\ \frac{4}{b}A_{-1}(z) &= \frac{(z - z_1)}{\pi} \arctan [\beta_1(z - z_1)] - \frac{(z - z_2)}{\pi} \arctan [\beta_2(z - z_2)] \\ &\quad - \frac{1}{2\pi\beta_1} \ln \left[1 + \beta_1^2(z - z_1)^2 \right] + \frac{1}{2\pi\beta_2} \ln \left[1 + \beta_2^2(z - z_2)^2 \right],\end{aligned}$$

The limit $\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2 \rightarrow +\infty$, when $z_1 = \bar{z}_1$ e $z_2 = \bar{z}_2$ with $z_1 < z_2$, corresponds to the pure mKdV case in the region $z < z_1$ and $z > z_2$, and pure sine-Gordon in the region $z_1 < z < z_2$.

Under such limiting case we have

$$\theta_1(z) = 1 - \frac{1}{2} \left(\frac{|z - z_1|}{(z - z_1)} - \frac{|z - z_2|}{(z - z_2)} \right), \quad \theta_2(z) = \frac{1}{2} \left(\frac{|z - z_1|}{(z - z_1)} - \frac{|z - z_2|}{(z - z_2)} \right),$$

such that

$$-\frac{1}{4a}A_3(z) = z - \frac{1}{2}(|z - z_1| - |z - z_2|), \quad \frac{4}{b}A_{-1}(z) = \frac{1}{2}(|z - z_1| - |z - z_2|).$$

Figure 4 represents the mKdV-SG-mKdV transition for one soliton solution.

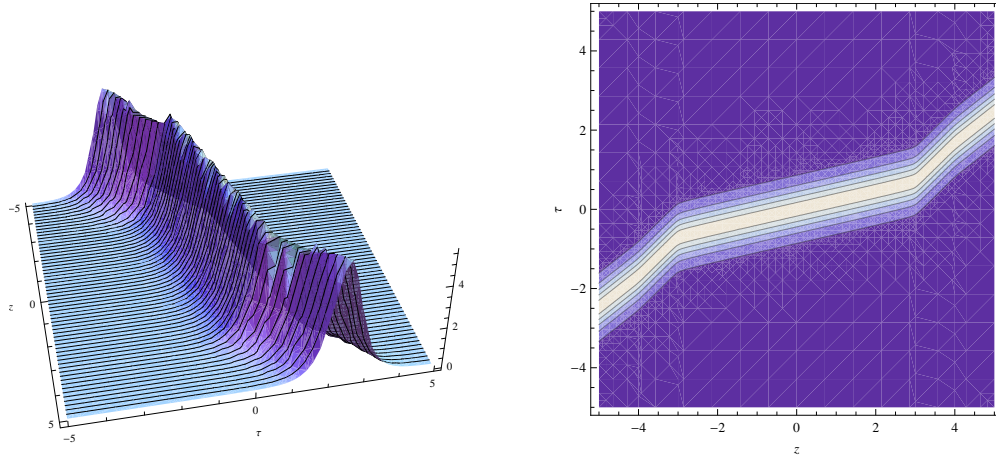


Figure 4: Plot (left) and contour plot (right) of ϕ_τ with $a = \frac{1}{10}$, $b = -\frac{7}{4}$, $\beta_1 = \beta_2 = \bar{\beta}_1 = \bar{\beta}_2 = 8\pi$, $z_1 = -4$, $z_2 = 4$, $\bar{z}_1 = -3$, $\bar{z}_2 = 3$ and $\gamma_1 = \frac{14}{10}$.

4.3 Two soliton solution

The two soliton solution ($\alpha_1 = \alpha_2 = 1$) can be represented by

As a conclusion, we have adapted the general dressing construction of soliton solutions to the mixed mKdV-SG hierarchy with arbitrary “time” dependent functions. The choice of such arbitrary functions as step-type functions allowed exact solutions describing smooth transitions from the mKdV to sine-Gordon regime and therefore more realistic models.

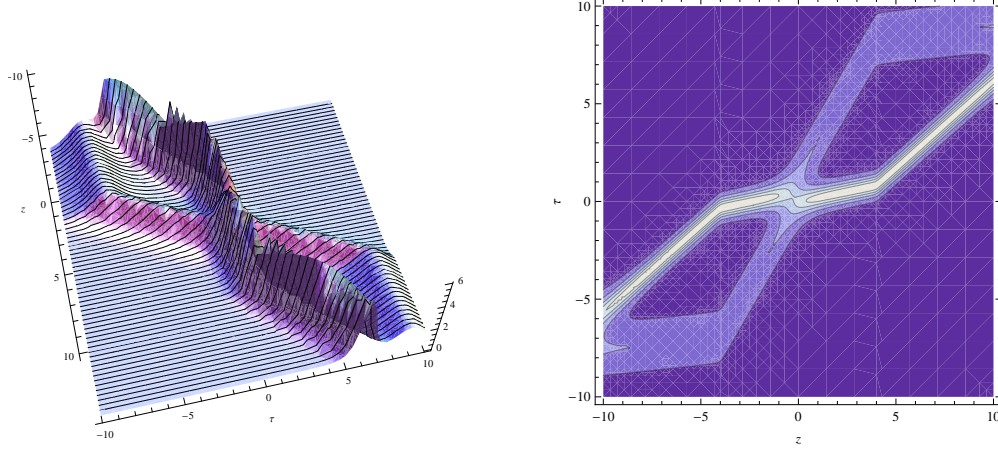


Figure 5: Plot (left) and contour plot (right) of ϕ_τ with $a = \frac{1}{10}$, $b = -\frac{7}{4}$, $\beta_1 = \beta_2 = \bar{\beta}_1 = \bar{\beta}_2 = 8\pi$, $z_1 = \bar{z}_1 = -4$, $z_2 = \bar{z}_2 = 4$, $\gamma_1 = \frac{3}{2}$ e $\gamma_2 = \frac{1}{2}$.

Acknowledgements

LHY and GRM acknowledges support from Fapesp and Capes respectively, JFG and AHZ thank CNPq for partial support.

References

- [1] K. Konno, W. Kameyama and H. Sanuki, *J. of Phys. Soc. of Japan* **37**(1974),171; K. Konno and H. Sanuki, *J. of Phys. Soc. of Japan* **37**(1974),292
- [2] H. Leblond and D. Mihalache, *Phys. Rev.* **A79** (2009) 063835
- [3] H. Leblond and F. Sanchez, *Phys. Rev.* **A67** (2003) 013804; H. Leblond, S.V. Sazonov, I.V. Mel'nikov, D. Mihalache and F. Sanchez, *Phys. Rev.* **A74** (2006) 063815
- [4] J.F. Gomes, G.R de Melo and A.H. Zimmerman, *J. Physics* **A42** (2009) 275208, arXiv:0903.0579 [nlin.SI]

- [5] A. Kundu, R. Sahadevan and L. Nalinidevi, *J. Physics* **A42** (2009) 115213, nlin-Si/0811.0924
- [6] K. Pradhan and P.K. Panigrahi, *J. Physics* **A39** (2006) L343
- [7] A. Kundu, *Phys. Rev.* **E79** (2009) 015601
- [8] O. Babelon and D. Bernard, *Int. J. Mod. Phys.* **A8** (1993) 507;
L. Ferreira, J.L. Miramontes and J. Sanchez-Guillén, *J. Math. Phys.* **38** (1997) 882,
arXiv:hep-th/9606066